

# A (very short) introduction to buildings

Brent Everitt\*

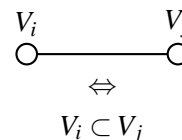
These lectures are an informal elementary introduction to buildings. They are written for, and by, a non-expert. The aim is to get to the definition of a building and feel that it is an entirely natural thing. To maintain the lecture style examples have replaced proofs. The notes at the end indicate where these proofs can be found.

The lectures are a distillation of the first few chapters of the books of Abramenko and Brown [AB08] and of Ronan [Ron09]. Lecture 1 illustrates all the features of a building in the context of an example, but without mentioning any building terminology. In principle anyone could read this. Lectures 2-4 firm-up and generalize these specifics: Coxeter groups appear in Lecture 2, chambers systems in Lecture 3 and the definition of a building in Lecture 4. Lecture 5 addresses where buildings come from by describing the first important example: the spherical building of an algebraic group.

## Lecture 1: The flag complex of a vector space

Let  $V$  be a 3-dimensional vector space over a field  $k$ . Let  $\Delta$  be the graph with vertices the non-trivial proper subspaces of  $V$ , and an edge connecting the vertices  $V_i$  and  $V_j$  whenever  $V_i$  is a subspace of  $V_j$  (as at right).

Figure 1 shows the graph  $\Delta$  when  $k$  is the field of order two. There are seven 1-dimensional subspaces – illustrated by the white vertices – and seven 2-dimensional subspaces (illustrated by the black vertices). Each 1-dimensional space is contained in three 2-dimensional spaces and each 2-dimensional space contains three 1-dimensional spaces. The duality here might remind the reader of projective geometry. Call the edges  $V_i \subset V_j$  of  $\Delta$  *chambers*.



Some more structure can be wrung out of this picture: there is an “ $\mathfrak{S}_3$ -valued metric”, with  $\mathfrak{S}_3$  the symmetric group, that gives the shortest route(s) through  $\Delta$  between any two chambers. To see how, suppose  $c, c'$  are chambers and we want a shortest route of edges connecting them:

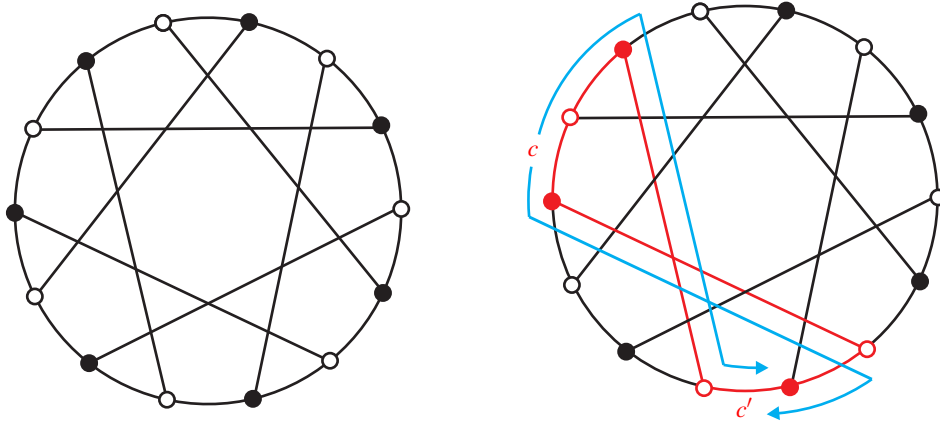
$$c = V_1 \subset V_2 \rightsquigarrow^{\text{shortest route}} c' = V'_1 \subset V'_2.$$

Make  $c$  and  $c'$  as different as possible by assuming that  $V_1 \neq V'_1$ ,  $V_2 \neq V'_2$  and  $V_2 \cap V'_2$  is a line different from  $V_1, V'_1$ . Changing notation, let  $L_1, L_2, L_3$  be lines with  $L_1 = V_1$ ,  $L_3 = V'_1$  and  $L_2 = V_2 \cap V'_2$ . One then gets  $V_2 = L_1 + L_2$  and  $V'_2 = L_2 + L_3$ .

We get a small piece of  $\Delta$ , a local picture containing  $c, c'$ , as below right. The field  $k$  wasn't mentioned at all in the previous paragraph, so this is the local picture for  $\Delta$  over any field. The

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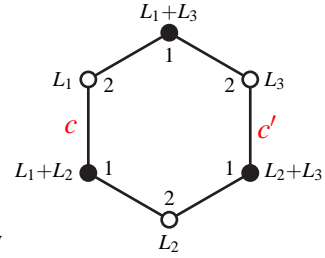
**Fig. 1.** The flag complex  $\Delta$  of the 3-dimensional vector space over  $k = \mathbb{F}_2$  (left) and a pair  $c, c'$  of chambers with the shortest routes  $s_1 s_2 s_1 = s_2 s_1 s_2 = (1, 3)$  between them in a local picture (right).

global picture however will be more complicated than the simple example given in Figure 1.

Say that chambers are *i*-incident when they differ only in the *i*-th position, so  $V_1 \subset V_2 \supset V'_1$ , ( $V_1 \neq V'_1$ ) are a pair of 1-incident chambers and  $V_2 \supset V_1 \subset V'_2$ , ( $V_2 \neq V'_2$ ) a pair of 2-incident chambers. Place the label *i* on a vertex of the local picture if the two chambers meeting at the vertex are *i*-incident.

The shortest routes from  $c$  to  $c'$  in the local picture are given by

$$c \xrightarrow{s_1 s_2 s_1} c'$$



where the route  $s_1 s_2 s_1$  means cross a 1-labeled vertex, then a 2-labeled vertex and then a 1-labeled vertex. Routes are read from left to right, although it obviously doesn't matter with the two above. These routes then take values in the symmetric group by letting  $s_1 = (1, 2)$  and  $s_2 = (2, 3)$ , so that both  $s_1 s_2 s_1$  and  $s_2 s_1 s_2$  give the permutation  $(1, 3) \in \mathfrak{S}_3$ . Define the  $\mathfrak{S}_3$ -distance between  $c, c'$  to be  $\delta(c, c') = (1, 3)$ .

For an arbitrary pair of chambers define  $\delta(c, c')$  to be the element of  $\mathfrak{S}_3$  obtained by situating the chambers  $c, c'$  in some local picture and taking the shortest route(s) – see Figure 1. The resulting map  $\delta : \Delta \times \Delta \rightarrow \mathfrak{S}_3$  can be thought of as a metric on  $\Delta$  taking values in  $\mathfrak{S}_3$ .

We will see in Lecture 4 why this map is well defined and doesn't depend on which local picture we choose containing  $c, c'$ , although an ad-hoc argument shows that an element of  $\mathfrak{S}_3$  can be associated in a canonical fashion to any pair of chambers. Take the  $c, c'$  above and write

$$c = 0 \subset L_1 \subset L_1 + L_2 \subset V = V_0 \subset V_1 \subset V_2 \subset V_3$$

and  $c' = V'_0 \subset \dots \subset V'_3$  similarly. For each *i* the filtration  $V_0 \subset V_1 \subset V_2 \subset V_3$  of  $V$  induces a filtration of the 1-dimensional quotient  $V'_i/V'_{i-1}$ :

$$(V'_i \cap V_0)/V'_{i-1} \subset \dots \subset (V'_i \cap V_3)/V'_{i-1}.$$

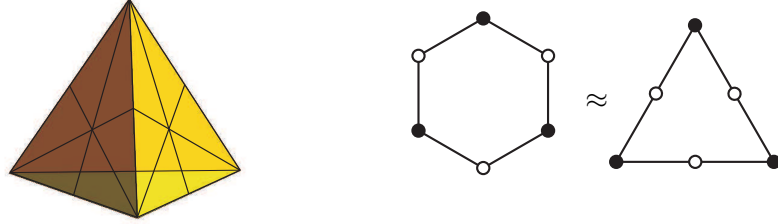
Any filtration of a 1-dimensional space must start with a sequence of trivial subspaces and end with a sequence of  $V'_i/V'_{i-1}$ 's. At some point in the middle the filtration jumps from being 0-dimensional to 1-dimensional:

<i>i</i>	$V'_i/V'_{i-1}$	filtration	“jump index” <i>j</i>
1	$L_3$	$0 \subset 0 \subset 0 \subset L_3$	3
2	$(L_2 + L_3)/L_3$	$0 \subset 0 \subset (L_2 + L_3)/L_3 \subset (L_2 + L_3)/L_3$	2
3	$V/L_2 + L_3$	$0 \subset V/L_2 + L_3 \subset V/L_2 + L_3 \subset V/L_2 + L_3$	1

Defining  $\pi(i) = j$  gives  $\pi = (1, 3) \in \mathfrak{S}_3$ . Summarizing:

*First rough definition of a building.* A building is a set of *chambers* with *i*-incidences between them, the *i* coming from some set *S*, together with a “*W*-valued metric” for *W* some group.

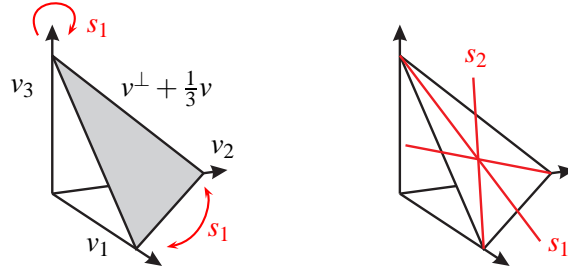
*Exercise.* Repeat the process with *V* 4-dimensional and  $\Delta$  a 2-dimensional simplicial complex having vertices the non-trivial subspaces of *V*, edges the pairs  $V_i \subset V_j$  and 2-cells the triples  $V_i \subset V_j \subset V_k$ . Show that the local picture is the boundary of a tetrahedron, barycentrically subdivided (the local picture in the previous case was the boundary of a triangle, barycentrically subdivided):



Returning to the running example, the symmetric group  $\mathfrak{S}_3$  is a reflection group, with the local picture and the resulting  $\delta$  coming from the geometry of these reflections. For suppose now that *V* is a Euclidean space – a real vector space with an inner product. Let  $v_1, v_2, v_3$  be an orthonormal basis and let  $\mathfrak{S}_3$  act on *V* by permuting coordinates:  $\pi \cdot v_i := v_{\pi(i)}$  for  $\pi \in \mathfrak{S}_3$ , and extend this action linearly to all of *V*. This action is not essential as the vector  $v = v_1 + v_2 + v_3$  is fixed by all  $\pi \in \mathfrak{S}_3$ . This can be gotten around by passing to the perp space

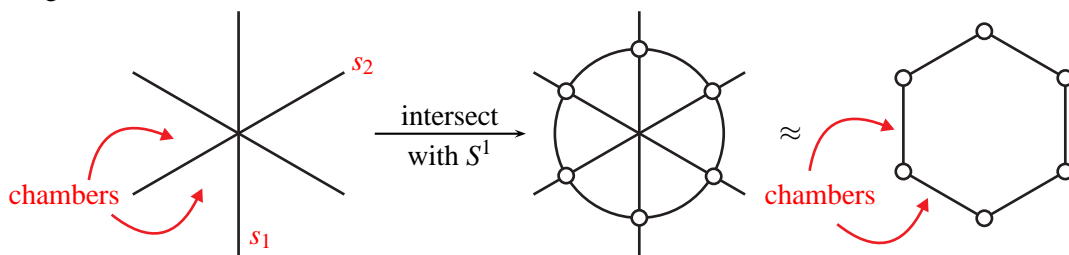
$$v^\perp = \{\sum \lambda_i v_i \mid \sum \lambda_i = 0\}.$$

The picture to keep in mind is the following, where  $v^\perp$  is translated off the origin to make it easier to see:



The element  $s_1 = (1, 2)$  acts as shown on the left, and this is the reflection in the plane with equation  $x_1 - x_2 = 0$ . Similarly  $s_2 = (2, 3)$  and  $(1, 3)$  are reflections in the planes  $x_2 - x_3 = 0$  and  $x_1 - x_3 = 0$ . These three planes chop  $v^\perp + \frac{1}{3}v$  into a triangle with its boundary barycentrically subdivided. So we start to see the local picture from the flag complex coming from the geometry of these reflecting hyperplanes.

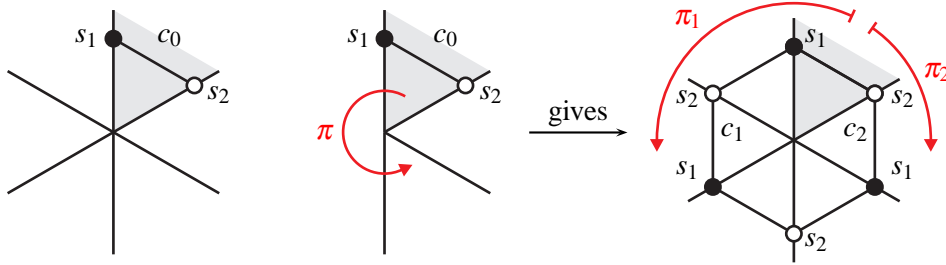
Putting  $v^\perp$  into the plane of the page decomposes the plane into six infinite wedge-shaped regions:



In the theory of reflection groups (Lecture 2) these regions are also called chambers. The chambers of our local picture are gotten back by intersecting the regions with the sphere  $S^1$ .

The  $\mathfrak{S}_3$ -action turns out to be regular on the chambers, i.e. given chambers  $c, c'$  there is a unique  $\pi \in \mathfrak{S}_3$  with  $\pi c = c'$ . This is most easily seen by brute force: fix a “fundamental” chamber  $c_0$  and show that the six elements of  $\mathfrak{S}_3$  send it to the six chambers in the decomposition above. In particular there is a one-one correspondence between the chambers and the elements of  $\mathfrak{S}_3$  given by  $\pi \in \mathfrak{S}_3 \leftrightarrow \text{chamber } \pi c_0$ .

This correspondence gives the incidence labelings of the hexagonal local picture: choose the fixed chamber  $c_0$  to be one of the two regions bounded by the reflecting lines for  $s_1$  and  $s_2$ . Starting with the edge of the hexagon contained in  $c_0$ , label its vertices by the corresponding reflections as below left:



Now transfer this labeled edge to the other chambers using the  $\mathfrak{S}_3$ -action as in the picture above middle; the result is shown above right, where the  $i$ 's have become  $s_i$ 's. That the two vertices on different sides of the same line have opposite labels is because the antipodal map  $x \mapsto -x$  is not in the action of  $\mathfrak{S}_3$  on the plane  $v^\perp$ .

Finally, to get the metric  $\delta$  observe that if  $c$  is some chamber of the local picture and  $\pi \in \mathfrak{S}_3$  sends  $c_0$  to  $c$ , then  $\pi = s_{i_1} \dots s_{i_k}$  where  $s_{i_k}, \dots, s_{i_1}$  are the labels (read from left to right) on the vertices crossed in a path in the hexagon from  $c_0$  to  $c$ . So for chambers  $c_1, c_2$  we have  $\delta(c_1, c_2) = \pi_1^{-1} \pi_2$  where  $c_i = \pi_i c_0$ . For our original  $c_1, c_2$  we have  $\pi_1 = s_1 s_2$ ,  $\pi_2 = s_2$ , hence  $\delta(c_1, c_2) = s_2 s_1 s_2$  as shown in the picture above.

*Second rough definition of building.* A building is a set of chambers with  $i$ -incidences, the  $i$  coming from some set  $S$ , together with a  $W$ -valued metric  $\delta$ , for  $W$  a reflection group and  $\delta$  arising from the geometry of  $W$ .

In the next three lectures we will make precise (and general) the ideas in this rough definition, but working in the reverse order: we start with reflection groups (Lecture 2), then an abstract version of chambers and incidences (Lecture 3) and finally  $W$ -valued metrics (Lecture 4).

## Lecture 2: Reflection Groups and Coxeter Groups

Reflection groups arise as the symmetries of familiar geometric objects; Coxeter groups are an abstraction of them. This lecture covers the basics. All vector spaces and linear maps here are over the reals  $\mathbb{R}$ .

Let  $V$  be a finite dimensional vector space. A *reflection* of  $V$  is a linear map  $s : V \rightarrow V$  for which there is a decomposition

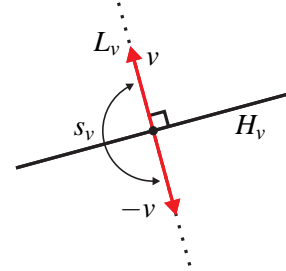
$$V = H_s \oplus L_s \quad (1)$$

where  $H_s$  is a hyperplane (a codimension 1 subspace);  $L_s$  is 1-dimensional; the restriction of  $s$  to  $H_s$  is the identity; and the restriction to  $L_s$  is the map  $x \mapsto -x$ . Thus a reflection fixes pointwise a mirror  $H_s$ , the reflecting hyperplane of  $s$ , and acts as multiplication by  $-1$  in some direction (the reflecting line) not lying in the mirror. In particular  $s$  is invertible.

A *reflection group*  $W$  is a subgroup of  $GL(V)$  generated by finitely many reflections.

*Example 1 (orthogonal reflections).* The most familiar examples of reflections are the orthogonal ones for which we further assume that  $V$  is a Euclidean space, i.e. is equipped with an inner product. Then  $s$  is orthogonal if in the decomposition (1) the line  $L_s = H_s^\perp$ , the orthogonal complement. In particular  $L_s$ , and hence the reflection, is determined by the reflecting hyperplane, unlike a general reflection where both the hyperplane and the line are needed.

If  $s$  is orthogonal then for any vector  $v$  in  $L_s$  we have  $s : v \mapsto -v$  with  $v^\perp$  fixed pointwise. Thus an orthogonal reflection  $s$  can be specified by just a non-zero vector  $v$ , as the reflection with  $H_s = v^\perp$  and  $L_s$  spanned by  $v$ . We write  $s = s_v$ ,  $H_s = H_v$ ,  $L_s = L_v$  and by choosing a sensible basis one gets that an orthogonal reflection is an orthogonal map of the Euclidean space.



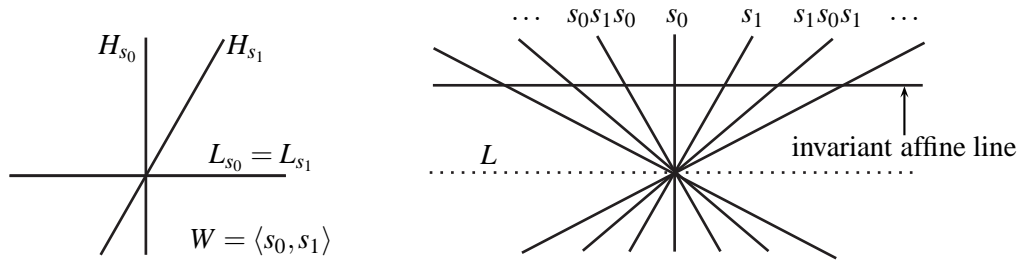
*Exercise.* Let  $\mathcal{H} = \{H_{v_1}, \dots, H_{v_m}\}$  be hyperplanes in Euclidean  $V$  and  $W$  the reflection group generated by the orthogonal reflections  $s_{v_1}, \dots, s_{v_m}$ . Show that if  $W\mathcal{H} = \mathcal{H}$ , i.e.  $gH_{v_i} = H_{v_j}$  for all  $g \in W$  and all  $v_i$ , then  $W$  is finite (*hint*:  $|W| \leq (2m)!$ ). It turns out (although this is harder) that  $\mathcal{H}$  then consists of all the reflecting hyperplanes of  $W$ .

*Example 2 (a finite reflection group).* Let  $V$  be Euclidean with orthonormal basis  $\{v_1, \dots, v_{n+1}\}$  and  $\mathcal{H}$  the hyperplanes  $H_{ij} := (v_i - v_j)^\perp$  for  $1 \leq i \neq j \leq n+1$  (in other words,  $H_{ij}$  is the hyperplane with equation  $x_i - x_j = 0$ ). The reflection  $s_{v_i - v_j}$  sends  $v_i - v_j$  to  $v_j - v_i$ , thus swapping the vectors  $v_i$  and  $v_j$ . Any other basis vector is orthogonal to  $v_i - v_j$ , so lies in  $H_{ij}$ , and is fixed by the reflection. Thus if  $\pi = (i, j) \in \mathfrak{S}_{n+1}$  then  $s_{v_i - v_j} H_{k\ell} = H_{\pi(k), \pi(\ell)}$ .

Now let  $W$  be the group generated by the reflections  $s_{v_i - v_j}$ . We have just shown that  $W\mathcal{H} = \mathcal{H}$ , so  $W$  is a finite reflection group by the Exercise above. Indeed,  $W$  is the symmetric group  $\mathfrak{S}_{n+1}$  acting by permuting coordinates as in Lecture 1. To make this identification we have already seen that each  $s_{v_i - v_j}$ , and so every element of  $W$ , permutes the basis vectors  $\{v_1, \dots, v_{n+1}\}$ . This gives a homomorphism  $W \rightarrow \mathfrak{S}_{n+1}$ . Injectivity of this homomorphism follows as the  $v_i$  span  $V$  and surjectivity as the transpositions  $(i, j)$  generate  $\mathfrak{S}_{n+1}$ .

The convex hull of the  $v_i$  is the standard  $n$ -simplex, barycentrically subdivided by its  $n(n-1)$  hyperplanes of reflectional symmetry, each of which is a reflecting hyperplane of  $W$ . This is the picture we had for  $n = 2$  in Lecture 1 (and  $n = 3$  in the Exercise). Finite reflection groups are often called *spherical* as their Coxeter complexes (the boundary of the barycentrically divided  $n$ -simplex in this case; see Example 7 for the general definition) are spheres.

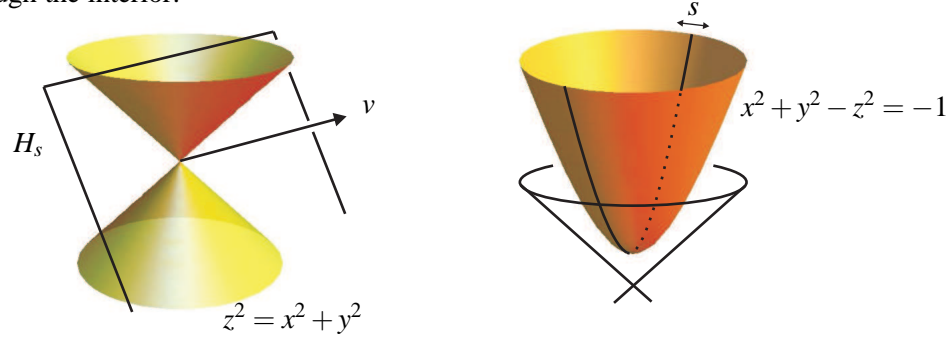
*Example 3 (An affine reflection group).* Let  $V$  be 2-dimensional and consider reflections  $s_0, s_1$  where the reflecting hyperplanes and lines are shown below left (there is no inner product this time). The reflecting hyperplanes are different but both have the same reflecting line:  $L_{s_0} = L = L_{s_1}$ . If  $W$  is the group generated by  $s_0, s_1$  then  $W$  leaves invariant any affine line parallel to  $L$  as the  $s_i$  do. But if  $\mathcal{H} = \{H_{s_0}, H_{s_1}\}$  then  $W\mathcal{H} \neq \mathcal{H}$  as  $s_0 H_{s_1} \notin \mathcal{H}$ . Indeed, we must expand  $\mathcal{H}$  to the infinite set shown below right before it becomes  $W$ -invariant:



In fact, by identifying the invariant affine line with the reals,  $W$  is isomorphic to the group of “affine” reflections of  $\mathbb{R}$  in the integers  $\mathbb{Z}$ , i.e. to the group of transformations of  $\mathbb{R}$  generated

by the maps  $s_n : x \mapsto 2n - x$  for  $n \in \mathbb{Z}$ . The element  $s_1 s_0$  acts on the affine line as the translation  $x \mapsto x + 2$  so has infinite order. In particular  $W$  is infinite. This also follows from  $\mathcal{H}$  being infinite as the reflections in the hyperplanes in  $\mathcal{H}$  are the  $W$ -conjugates of  $s_0, s_1$ .

*Example 4 (hyperbolic reflections).* Let  $V$  be 3-dimensional and again there is no inner product. Let  $a, b, c$  be real numbers such that  $a^2 + b^2 > c^2$ , and consider the reflection  $s$  with reflecting hyperplane  $H_s$  having the equation  $ax + by - cz = 0$  and reflecting line  $L_s$  spanned by the vector  $v = (a, b, c)$ . Then  $v$  lies on the outside of the pair of cones with equation  $z^2 = x^2 + y^2$  and  $H_s$  passes through the interior:



One can check that  $s$  leaves invariant each sheet of the two sheeted hyperboloid with equation  $x^2 + y^2 - z^2 = -1$ . Either sheet is a model for the hyperbolic plane. Intersecting  $H_s$  with the top sheet gives a hyperbola – a straight line of hyperbolic geometry – and  $s$  is the “hyperbolic reflection” of the plane in this line<sup>1</sup>.

Returning to the finite orthogonal case, let  $V$  be Euclidean,  $\mathcal{H} = \{H_i\}_{i \in T}$  a finite set of hyperplanes and  $W = \langle s_i \rangle_{i \in T}$  the group generated by the orthogonal reflections in the  $H_i$ . Suppose also that  $W\mathcal{H} = \mathcal{H}$ , so  $W$  is finite and  $\mathcal{H}$  is the set of all reflecting hyperplanes of  $W$  as in the Exercise above.

For each  $i \in T$  choose a linear functional  $\alpha_i \in V^*$  with  $H_i = \ker \alpha_i$ . The choice of  $\alpha_i$  is unique upto scalar multiple and  $H_i$  consists of those  $v \in V$  with  $\alpha_i(v) = 0$ . The two sides (or half-spaces) of the hyperplane consist of the  $v$  with  $\alpha_i(v) > 0$  or the  $v$  with  $\alpha_i(v) < 0$ .

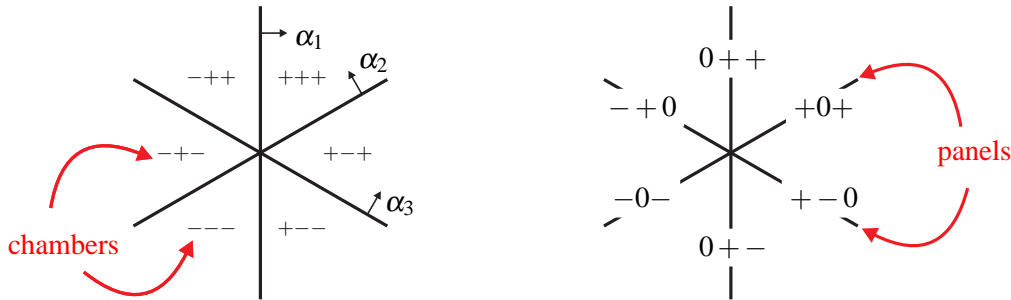
Fix an  $T$ -tuple  $\varepsilon = (\varepsilon_i)_{i \in T}$  with  $\varepsilon_i \in \{\pm 1\}$ . Consider the set

$$c = c(\varepsilon) = \{v \in V \mid \varepsilon_i \alpha_i(v) > 0 \text{ for all } i\}. \quad (2)$$

So each  $\alpha_i(v)$  is non-zero and  $\alpha_i(v), \varepsilon_i$  have the same sign for all  $i$ . If this set is non-empty then call it a *chamber* of  $W$ . A non-empty set of the form

$$a = a(\varepsilon) = \{v \in V \mid \alpha_{i_0}(v) = 0 \text{ for some } i_0, \text{ and } \varepsilon_i \alpha_i(v) > 0 \text{ for all } i \neq i_0\} \quad (3)$$

is called a *panel*. Here is the example from Lecture 1:



<sup>1</sup> Although there is no inner product in Examples 3 and 4, it is possible to endow  $V$  with a bilinear form so that the reflections are “orthogonal” with respect to this form.

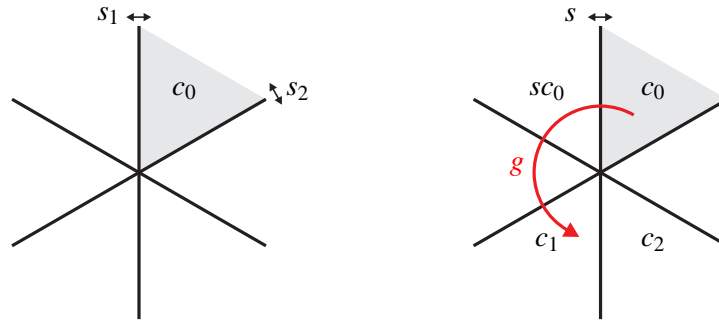
where there are three hyperplanes in  $\mathcal{H}$  and the  $\alpha_i$  are chosen so that  $\alpha_i(v) > 0$  for those  $v$  on the side indicated by the arrow. The chambers are marked by their  $T$ -tuples. There are  $2^3 = 8$   $T$ -tuples but only 6 chambers because the tuples  $++-$  and  $--+$  give empty sets in (2). Extend the notation to include panels (3) by placing a 0 in the  $i_0$ -th position. There are then  $3 \cdot 2^2 = 12$  such tuples but only 6 give non-empty panels, with two lying on each reflecting line.

There is an obvious notion of adjacency between chambers suggested by these pictures. Say that  $a$  is a panel of the chamber  $c$  if the corresponding  $T$ -tuples are identical except in one position where the tuple for  $a$  has a 0. It turns out that this can also be defined topologically:  $a$  is a panel of  $c$  exactly when  $\bar{a} \subset \bar{c}$ , the closures of these sets with respect to the usual topology on  $V$ .

We then say that chambers  $c_1$  and  $c_2$  are *adjacent* if they share a common panel. In the Example from Lecture 1, chambers are adjacent when they share a common edge.

The adjacency relation can be refined by bringing the reflection group  $W$  into the picture. In Lecture 1 we saw that  $\mathfrak{S}_3$  acts regularly as a reflection group on the chambers. This turns out to be true in general for the  $W$ -action on the chambers: given chambers  $c, c'$  there is a unique  $g \in W$  with  $gc = c'$ . Fix one of the chambers  $c_0$ . Then the regular action gives that the chambers are in one-one correspondence with the elements of  $W$  via  $g \in W \leftrightarrow \text{chamber } gc_0$ .

Now let  $S = \{s_1, \dots, s_n\}$  be those reflections in  $W$  whose hyperplanes  $H_1, \dots, H_n$  are spanned by a panel of the fixed chamber  $c_0$ . Thus  $S = \{s_1, s_2\}$  for the  $c_0$  in the example from Lecture 1:



Suppose  $c_1, c_2$  are a pair of adjacent chambers as above right. Then there is a  $g \in W$  with  $c_1 = gc_0$ . Translating the picture back to  $c_0$  we have  $g^{-1}c_1 = c_0$  and  $g^{-1}c_2$  are adjacent chambers, and the common panel of  $c_1, c_2$  is sent by  $g^{-1}$  to a common panel of  $c_0$  and  $g^{-1}c_2$  (these are most easily seen using the topological version of adjacency). If  $s \in S$  is the reflection in the hyperplane spanned by the common panel of  $c_0$  and  $g^{-1}c_2$ , then the chamber  $g^{-1}c_2$  is the same as the chamber  $sc_0$ .

Thus  $c_1 = gc_0$ ,  $c_2 = (gs)c_0$ , and we have the following more refined description of adjacency:

$$\text{the chambers adjacent to the chamber } gc_0 \text{ are the } (gs)c_0 \text{ for } s \in S. \quad (4)$$

When  $S = \{s_1, \dots, s_n\}$  we say that chambers  $gc_0$  and  $gs_i c_0$  are  $i$ -incident. In our running example, the chambers adjacent to  $gc_0$  are  $gs_1 c_0$  and  $gs_2 c_0$ , and these are the two that were 1- and 2-incident to  $gc_0$  in Lecture 1.

*Coxeter groups.* We motivate the definition of Coxeter group by quoting two facts, staying with the assumptions above where  $W$  is generated by orthogonal reflections in finitely many hyperplanes  $\mathcal{H}$  with  $W\mathcal{H} = \mathcal{H}$ :

*Fact 1.* The group  $W$  is generated by the reflections  $s \in S$  in those hyperplanes spanned by a panel of the fixed chamber  $c_0$ .



In our running example we can see how a proof might work using induction on the “distance” of a chamber from  $c_0$ . If  $g$  is an element of  $W$  then there is a chamber adjacent to the chamber  $gc_0$  that is closer to  $c_0$  than  $gc_0$  is. If this closer chamber is  $g'c_0$  say, then by (4) we have  $g = g's$  for some  $s \in S$ . Repeat the process until  $g$  completely decomposes as a word in the  $s \in S$ .

**Fact 2.** With respect to the generators  $S$  the group  $W$  admits a presentation

$$\langle s \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle \quad (5)$$

where the  $m_{ij} \in \mathbb{Z}^{\geq 1}$  and are such that  $m_{ij} = m_{ji}$ , and  $m_{ij} = 1$  if and only if  $i = j$  (so in particular,  $s_i^2 = 1$ ).

If  $s_i$  and  $s_j$  are reflections in  $W$  finite, then the element  $s_i s_j$  has finite order  $m_{ij} \geq 2$ . So the relations in the presentation (5) certainly hold. The content of Fact 2 is that these relations suffice. Geometrically,  $s_i s_j$  is a rotation “about” the intersection  $H_i \cap H_j$  of the corresponding hyperplanes.

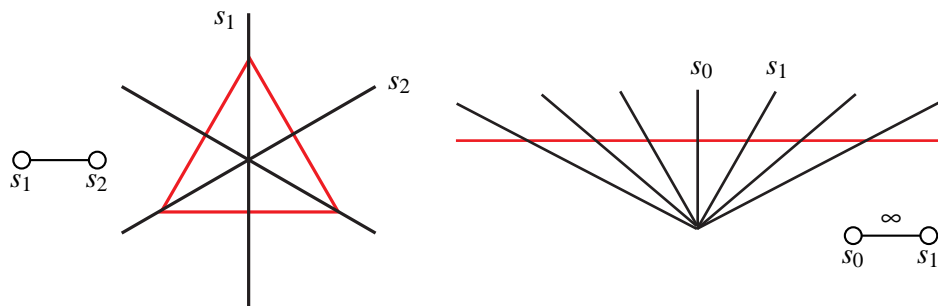
In our running example of the action of  $\mathfrak{S}_3$  on 3-dimensional  $V$  we have  $S = \{s_1, s_2\}$  with  $s_1 s_2$  of order 3 – a  $1/3$ -turn anticlockwise of the hexagon. The presentation (5) of  $\mathfrak{S}_3$  is thus  $\langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 \rangle$  with  $s_1 = (1, 2)$ ,  $s_2 = (2, 3)$  and  $s_1 s_2 = (1, 2, 3)$ .

Here is the promised abstraction of reflection group: a group  $W$  is called a *Coxeter group* if it admits a presentation (5) with respect to some finite  $S$ , where the  $m_{ij} \in \mathbb{Z}^{\geq 1} \cup \{\infty\}$  satisfy the rules following (5). To emphasize the dependence of the notion on the choice of generators we will write  $(W, S)$  for a Coxeter group.

We want the new concept to cover all the examples we have seen so far in this lecture, including the affine group in Example 3 where the element  $s_1 s_0$  had infinite order. This is why in the definition of Coxeter group the conditions on the  $m_{ij}$  are relaxed to allow them to be infinite. A relation  $(s_i s_j)^{m_{ij}} = 1$  is omitted from the presentation when  $m_{ij} = \infty$ .

There is a standard shorthand for a Coxeter presentation (5) called the Coxeter symbol. This is a graph with nodes the  $s_i \in S$ , and where nodes  $s_i$  and  $s_j$  are joined by an edge labeled  $m_{ij}$  if  $m_{ij} \geq 4$ , an unlabeled edge if  $m_{ij} = 3$  and no edge when  $m_{ij} = 2$  (as above right).

The examples from Lecture 1 and Example 3 are then:



**Remark.** What is the relationship between the concrete reflection groups defined at the beginning of this lecture and the abstract Coxeter groups defined at the end? The answer is that the Coxeter groups are the *discrete* reflection groups in the following sense: for a Coxeter group  $(W, S)$  one can construct a faithful representation  $(W, S) \rightarrow GL(V)$  for some vector space  $V$ , where the  $s \in S$  act on  $V$  as reflections. Moreover, the image of  $(W, S)$  is a discrete subgroup of  $GL(V)$ . Conversely a reflection group  $W$  that is a discrete subgroup of  $GL(V)$  is a Coxeter group with  $S$  some subset of the reflections. In general a (non-discrete) reflection group is not a Coxeter group.



### Lecture 3: Chamber Systems and Coxeter Complexes

We have seen several examples of sets of chambers with different kinds of incidences between them. This lecture introduces the formalization of this idea: chamber systems.

A *chamber system* over a finite set  $I$  is a set  $\Delta$  equipped with equivalence relations  $\sim_i$ , one for each  $i \in I$ . The  $c \in \Delta$  are the *chambers* and two chambers are *i*-incident when  $c \sim_i c'$ .

The generic picture to keep in mind is on the right where chambers are *i*-incident if they share a common *i*-labeled edge.

Thus,  $c_0 \sim_1 c_1, c_0 \sim_2 c_2$ , etc.

A *gallery* in a chamber system  $\Delta$  is a sequence of chambers

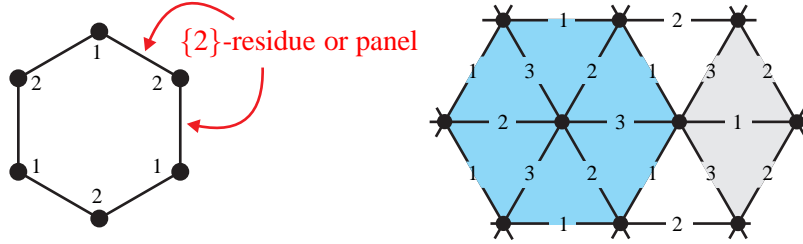
$$c_0 \sim_{i_1} c_1 \sim_{i_2} \cdots \sim_{i_k} c_k \quad (6)$$

with  $c_{j-1}$  and  $c_j$   $i_j$ -incident and  $c_j \neq c_{j+1}$ . The last condition is a technicality to help with the accounting. We say that the gallery (6) has type  $i_1 i_2 \dots i_k$ , and write  $c_0 \rightarrow_f c_k$  where  $f = i_1 i_2 \dots i_k$ . If  $J \subseteq I$  then a *J-gallery* is a gallery of type  $i_1 i_2 \dots i_k$  with the  $i_j \in J$ .

A subset  $\Delta' \subseteq \Delta$  of chambers is *J-connected* when any two  $c, c' \in \Delta'$  can be joined by a *J-gallery* that is contained in  $\Delta'$ . The *J-residues* of  $\Delta$  are the *J*-connected components and they have rank  $|J|$ . Thus the chambers themselves are the rank 0 residues. The rank 1 residues are the equivalence classes of the equivalence relations  $\sim_i$  as  $i$  runs through  $I$ . Call these rank 1 residues the *panels* of  $\Delta$ . The chamber system itself has rank  $|I|$ .

A *morphism*  $\alpha : \Delta \rightarrow \Delta'$  of chamber systems (both over the same set  $I$ ) is a map of the chambers of  $\Delta$  to the chambers of  $\Delta'$  that preserves *i*-incidence for all  $i$ : if  $c \sim_i c'$  in  $\Delta$  then  $\alpha(c) \sim_i \alpha(c')$  in  $\Delta'$ . An *isomorphism* is a bijective morphism whose inverse is also a morphism.

*Example 5.* The local picture from Lecture 1 (below left) is a chamber system over  $I = \{1, 2\}$ , with chambers the edges, and two chambers *i*-incident when they share a common *i*-labeled vertex. The  $\{i\}$ -residues, or panels, are the pairs of edges having a *i*-labeled vertex in common; in particular each panel contains exactly two chambers and there is a one-one correspondence between the panels and the vertices:



The example above right has chambers the 2-simplices,  $I = \{1, 2, 3\}$ , and two chambers *i*-incident when they share a common *i*-labeled edge. The blue 2-simplices are a  $\{2, 3\}$ -residue and the gray 2-simplices a  $\{1\}$ -residue or panel (so again, each panel contains two chambers). The six chambers in the rank 2 residue have a single common vertex at their center, and there is a one-one correspondence between the rank 2 residues and the vertices; similarly there is a one-one correspondence between the panels and the edges. So the chambers are the maximal dimensional simplices and the residues correspond to the lower dimensional ones. We will return to this point below.

*Example 6 (flag complexes).* Generalizing the example of Lecture 1, let  $V$  be an  $n$ -dimensional vector space over a field  $k$ . A (maximal) *flag* is a sequence of subspaces  $V_1 \subset \cdots \subset V_{n-1}$  with  $\dim V_i = i$ . Let  $\Delta$  be the chamber system over  $I = \{1, \dots, n-1\}$  whose chambers are the flags and where

$$(V_1 \subset \cdots \subset V_{n-1}) \sim_i (V'_1 \subset \cdots \subset V'_{n-1})$$

when  $V_j = V'_j$  for  $j \neq i$ , i.e. the flags differ only in the  $i$ -th position. The chambers in the panel (or  $\{i\}$ -residue) containing  $V_1 \subset \cdots \subset V_{n-1}$  correspond to the 1-dimensional subspaces of the 2-dimensional space  $V_{i+1}/V_{i-1}$ . If  $k$  is finite of order  $q$  then each panel thus contains  $q + 1$  chambers; if  $k$  is infinite then each panel contains infinitely many chambers.

*Example 7 (Coxeter complexes).* In Lecture 2 we defined chambers, panels and  $i$ -incidence for a finite reflection group  $W$  acting on a Euclidean space: the chambers were in one-one correspondence with the elements of  $W$  via  $g \leftrightarrow gc_0$  ( $c_0$  a fixed fundamental chamber), and  $gc_0$  and  $g'c_0$  were  $i$ -incident when  $g' = gs_i$ .

Now let  $(W, S)$  be a Coxeter group with  $S = \{s_i\}_{i \in I}$ . The *Coxeter complex*  $\Delta_W$  is the chamber system over  $I$  with chambers the elements of  $W$  and

$$g \sim_i g' \text{ if and only if } g' = gs_i \text{ in } W. \quad (7)$$

Thus  $g \sim_i gs_i$  and also  $gs_i \sim_i gs_is_i = g$ . The  $\{i\}$ -panel containing  $g$  is thus  $\{g, gs_i\}$ , so each panel contains exactly two chambers (which can be thought of as lying on either side of the panel). This is the picture the geometry was giving us in Lecture 2. A gallery in  $\Delta_W$  has the form

$$g \sim_{i_1} gs_{i_1} \sim_{i_2} gs_{i_1}s_{i_2} \sim \cdots \sim_{i_k} gs_{i_1}s_{i_2} \cdots s_{i_k}.$$

If  $f = i_1 i_2 \cdots i_k$  and  $s_f = s_{i_1} s_{i_2} \cdots s_{i_k}$ , then there is a gallery  $g \rightarrow_f g'$  in  $\Delta_W$  exactly when  $g' = gs_f$  in  $W$ .

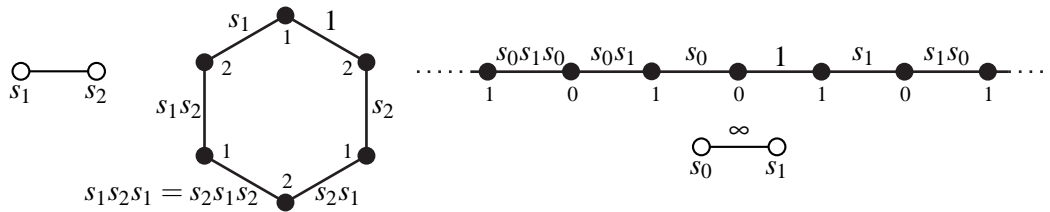
If  $s_i, s_j \in S$  then starting at the chamber  $g$  we can set off in the two directions given by the galleries:

$$g \sim_i gs_i \sim_j gs_is_j \sim_i gs_is_js_i \cdots \quad \text{and} \quad g \sim_j gs_j \sim_i gs_js_i \sim_j gs_js_is_j \cdots$$

If the order of  $s_i s_j$  is finite, then  $(s_i s_j)^{m_{ij}} = 1$  is equivalent to the relation

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots,$$

where there are  $m_{ij}$  symbols on both sides, so the two galleries above, despite starting out in opposite directions, nevertheless end up at the same place: namely the chamber  $gs_i s_j s_i \cdots = gs_j s_i s_j \cdots$ . Thus the  $\{i, j\}$ -residues in  $\Delta_W$  are circuits containing  $2m_{ij}$  chambers when  $s_i s_j$  has finite order. If the order is not finite then the residue is an infinitely long line of chambers stretching in “both directions” from  $g$ . The two Coxeter groups from the end of Lecture 2 have Coxeter complexes illustrating both these phenomena:



*Aside.* In all our pictures of chamber systems, the chambers, panels and lower dimensional cells have been simplices. It turns out that chamber systems are particularly nice examples of simplicial complexes where the chambers are the maximal dimensional simplices. Moreover in all the chamber systems arising in these lectures there is a correspondence between the lower dimensional simplices and the residues.

Recall that an abstract simplicial complex  $X$  with vertex set  $V$  is a collection of subsets of  $V$  such that

- (a).  $\sigma \in X$  and  $\tau \subset \sigma \Rightarrow \tau \in X$  and (b).  $\{v\} \in X$  for all  $v \in V$ .

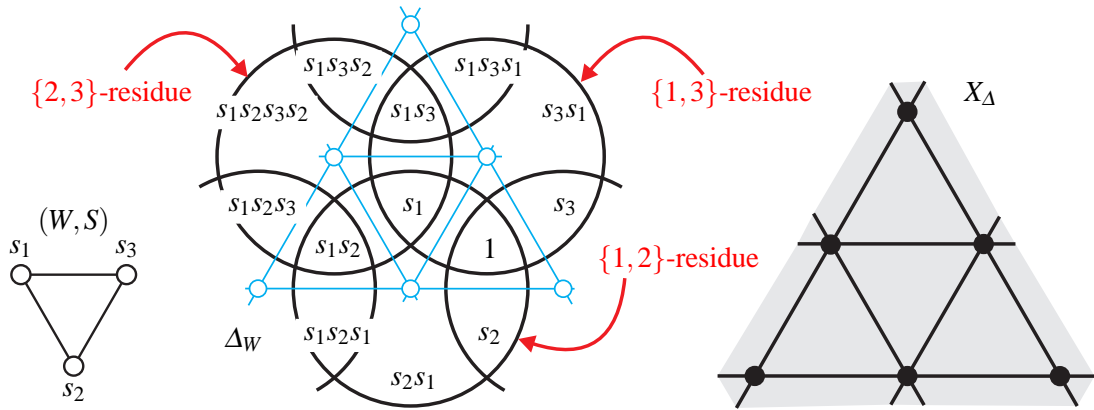
A  $\sigma = \{v_0, \dots, v_k\}$  is a  $k$ -simplex of the simplicial complex  $X$ . The empty set  $\emptyset$  is by convention the unique simplex of dimension  $-1$ .

Now let  $\Delta$  be a chamber system over  $I$  and let  $V$  be the set of residues of rank  $|I| - 1$  (recall that there is only one residue of rank  $|I|$ , namely  $\Delta$  itself). Then let  $X_\Delta$  be the simplicial complex with vertex set  $V$  such that if  $R_0, \dots, R_k$  are rank  $|I| - 1$  residues then

$$\sigma = \{R_0, \dots, R_k\} \text{ is a } k\text{-simplex of } X_\Delta \Leftrightarrow \bigcap R_i \neq \emptyset.$$

In other words,  $X_\Delta$  is the nerve of the covering of  $\Delta$  by rank  $|I| - 1$  residues. Take the empty intersection to be the union  $\bigcup V R_i$ , and observe that the maximum dimension a simplex can have is  $|I| - 1$ .

We illustrate with the Coxeter complex  $\Delta_W$  of the Coxeter group  $(W, S)$  with the symbol shown below left. Some elements of  $W$  have been written down in a suggestive pattern, grouped into three rank 2 residues. The simplicial complex  $X_\Delta$  acquires a 2-simplex from these residues as any two intersect in a residue of rank 1 and all three intersect in a residue of rank 0. In fact  $X_W$  is the infinite tiling of the plane from Example 5:



It would seem from this example that if  $R_0, \dots, R_k$  are rank  $|I| - 1$  residues over  $J_0, \dots, J_k$  with  $\bigcap R_i \neq \emptyset$ , then  $\bigcap R_i$  (if it is non empty) is a residue over  $\bigcap J_i$ . In fact this is always true for a Coxeter complex and indeed any building, although not for an arbitrary chamber system. As  $\bigcap J_i$  has  $|I| - (k + 1)$  elements, there is a one-one correspondence between the simplices of  $X_\Delta$  and the residues of  $\Delta$ :

$$\text{codimension } \ell \text{ simplices } \sigma = \{R_0, \dots, R_m\} \leftrightarrow \text{residues } \bigcap_{i=0}^m R_i \text{ of rank } \ell,$$

where  $m = |I| - (\ell + 1)$ . So for buildings the chambers of a chamber system  $\Delta$  are the top dimensional simplices of  $X_\Delta$ , with the lower dimensional simplices given by the residues.

We now have all the properties of chamber complexes that we need. We finish the lecture by defining a  $W$ -valued metric on the Coxeter complex  $\Delta_W$ .

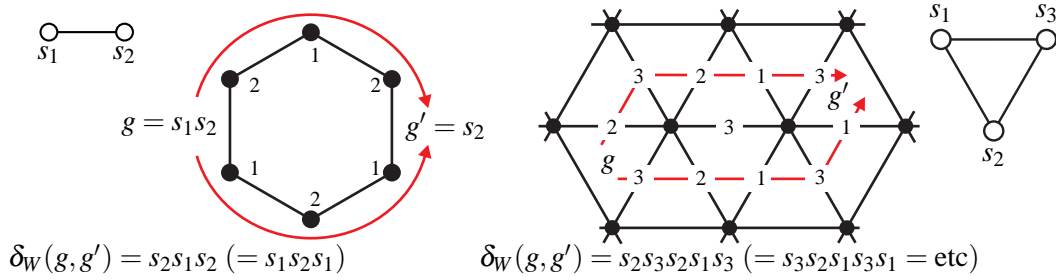
If  $(W, S)$  is a Coxeter group and  $f = i_1 i_2 \dots i_k$  with  $s_f = s_{i_1} s_{i_2} \dots s_{i_k}$ , then there is a gallery  $g \rightarrow_f g'$  in  $\Delta_W$  exactly when  $g' = g s_f$  in  $W$ . Call such a gallery *minimal* if there is no gallery in  $\Delta_W$  from  $g$  to  $g'$  that passes through fewer chambers. Call an expression  $s_f = s_{i_1} s_{i_2} \dots s_{i_k}$  *reduced* if there is no expression for  $s_f$  involving fewer  $s$ 's (counted with multiplicity). Thus a gallery  $g \rightarrow_f g'$  is minimal if and only if the expression  $s_f$  is reduced.

Define  $\delta_W : \Delta_W \times \Delta_W \rightarrow W$  by  $\delta_W(g, g') = g^{-1} g'$ . Then

$$\delta_W(g, g') = s_f \Leftrightarrow g' = g s_f \Leftrightarrow \text{there is a gallery } g \rightarrow_f g'. \quad (8)$$

Moreover,  $\delta_W(g, g')$  is reduced if and only if the gallery  $g \rightarrow_f g'$  is minimal. A slight relaxation will define the metric on an arbitrary building.

We end with two examples, one of which is our running one:



#### Lecture 4: Buildings and Apartments

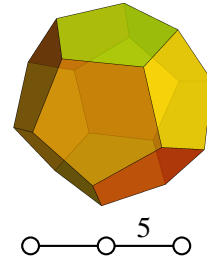
Let  $(W, S)$  be a Coxeter group with  $S = \{s_i\}_{i \in I}$ . A *building of type  $(W, S)$*  is a chamber system  $\Delta$  over  $I$  such that:

- (B1). every panel of  $\Delta$  contains at least two chambers;
- (B2).  $\Delta$  has a  $W$ -valued metric  $\delta : \Delta \times \Delta \rightarrow W$  such that if  $s_f = s_{i_1} \dots s_{i_k}$  is a *reduced* expression in  $W$  then

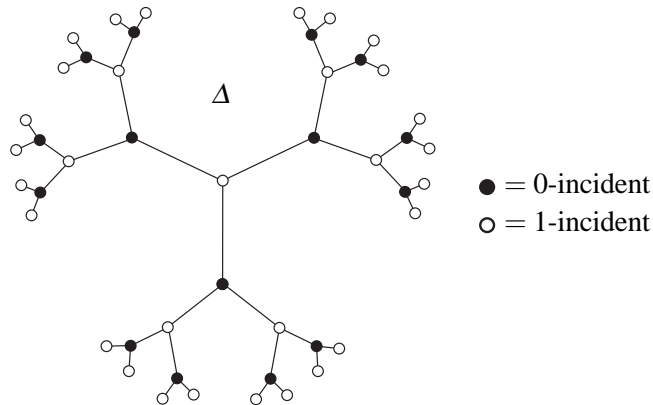
$$\delta(c, c') = s_f \Leftrightarrow \text{there is a gallery } c \rightarrow_f c' \text{ in } \Delta.$$

*Example 8 (Coxeter complexes).* There is at least one building for every Coxeter group  $(W, S)$ , namely the Coxeter complex  $\Delta_W$  with  $\delta = \delta_W$  in (8), hence (B2). For (B1) we observed in Example 7 that the panels in  $\Delta_W$  have the form  $\{g, gs\}$  for  $g \in W$  and  $s \in S$ . Such a building, where each panel has the minimum possible number of chambers, is said to be *thin*. It turns out that the thin buildings are precisely the Coxeter complexes.

There are quite naturally arising Coxeter groups for which the Coxeter complex is the *only* finite building. The group with symbol at right, the reflectional symmetries of a regular dodecahedron, is an example.

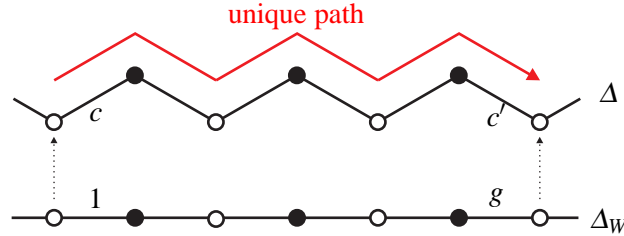


*Example 9 (an affine building).* An affine building is a building where the Coxeter group  $(W, S)$  is an affine reflection group as in Example 3. Take this example, with  $S = \{s_0, s_1\}$  and Coxeter symbol  $\bigcirc \overset{\infty}{\text{---}} \bigcirc$ . Let  $\Delta$  be the chamber system over  $I = \{0, 1\}$  shown below – an infinite 3-valent tree. The edges are the chambers, and two chambers are 0-incident when they share a common black vertex and 1-incident when they share a common white vertex.



Each panel thus contains three chambers, hence (B1). The Coxeter complex  $\Delta_W$  is in Example 7 (also a tree).

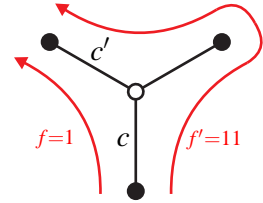
To define the  $W$ -metric on  $\Delta$  recall that in a tree there is a unique path between edges without “backtracking”: a backtrack is a path that crosses an edge and then immediately comes back across the edge again. For chambers  $c, c' \in \Delta$ , match this unique path between  $c$  and  $c'$  with the same path starting at 1 in  $\Delta_W$ :



and define  $\delta(c, c')$  to be the resulting  $g$ . To see (B2), let  $\delta(c, c') = g \in W$  and suppose that  $g = s_{j_1} \dots s_{j_\ell}$ . Then by (8) there is a gallery in  $\Delta_W$  from 1 to  $g$  and of type  $j_1 \dots j_\ell$ . As  $\Delta_W$  is also a tree this gallery differs from the unique minimal one only by backtracks. First transfer this minimal gallery to  $\Delta$  to get the minimal gallery from  $c$  to  $c'$ , and then transfer the backtracks to obtain a gallery of type  $j_1 \dots j_\ell$  from  $c$  to  $c'$ . Conversely if there is a gallery from  $c$  to  $c'$  of type  $j_1 \dots j_\ell$  with  $s_{j_1} \dots s_{j_\ell}$  *reduced*, then in particular no two consecutive  $s$ 's are the same and so the gallery has no backtracks. Thus it is *the* unique minimal gallery from  $c$  to  $c'$  giving  $\delta(c, c') = s_{j_1} \dots s_{j_\ell}$  by definition.

In a Coxeter complex we have  $\delta_W(c, c') = s_{i_1} \dots s_{i_k}$  if and only if there is a gallery of type  $i_1 \dots i_k$  from  $c$  to  $c'$ , but in an arbitrary building there is this extra condition that the word  $s_{i_1} \dots s_{i_k}$  be reduced. We can see why in the example above: if there is a gallery of type  $i_1 \dots i_k$  from  $c$  to  $c'$  with  $s_{i_1} \dots s_{i_k}$  not reduced, then  $\delta(c, c')$  need not necessarily be  $s_{i_1} \dots s_{i_k}$ . For example, if we have three incident chambers as at right then there is a gallery of type 1 from  $c$  to  $c'$  with  $s_1$  reduced, hence  $\delta(c, c') = s_1$ . The non-reduced gallery  $c \rightarrow_{11} c'$  does not give  $\delta(c, c') = s_1 s_1$ , as  $s_1 s_1 = 1 \neq s_1$ .

Example 9 is our first of a *thick* building: one where every panel contains at least three chambers. “Thick” is generally taken to be synonymous with interesting.



In the example of Lecture 1 (as well as Example 9) we defined the  $W$ -metric  $\delta$  by situating a pair of chambers  $c, c'$  inside a copy of the Coxeter complex  $\Delta_W$  and transferring the metric  $\delta_W$  defined in (8). We need to see that this process is well defined – although this is obvious in Example 9 – and that the resulting  $\delta$  satisfies (B2). This leads to an alternative definition of building (Theorem 2 below) based on this idea of defining  $\delta$  locally.

Let  $(\Delta, \delta)$  and  $(\Delta', \delta')$  be buildings of type  $(W, S)$  and  $X \subset (\Delta, \delta), Y \subset (\Delta', \delta')$  be subsets. A morphism  $\alpha : X \rightarrow Y$  is an *isometry* when it preserves the  $W$ -metrics: for all chambers  $c, c'$  in  $X$  we have  $\delta'(\alpha(c), \alpha(c')) = \delta(c, c')$ . A simple example is if  $g_0 \in W$ , then  $g \mapsto g_0 g$  is an isometry  $\Delta_W \rightarrow \Delta_W$ .

The following result guarantees the existence of copies of the Coxeter complex in a building:

**Theorem 1.** *Let  $\Delta$  be a building of type  $(W, S)$  and  $X$  a subset of the Coxeter complex  $\Delta_W$ . Then any isometry  $X \rightarrow \Delta$  extends to an isometry  $\Delta_W \rightarrow \Delta$ .*

An *apartment* in a building  $\Delta$  of type  $(W, S)$  is an isometric image of the Coxeter complex  $\Delta_W$ , i.e. a subset of the form  $\alpha(\Delta_W)$  for  $\alpha : \Delta_W \rightarrow \Delta$  some isometry. Apartments are the local pictures we saw in Lecture 1.

We are particularly interested in the following two consequences of Theorem 1:

Any two chambers  $c, c'$  lie in some apartment  $A$ . (9)

(If  $\delta(c, c') = g \in W$ , then  $X = (1, g) \subset \Delta_W \mapsto (c, c') \subset \Delta$  is an isometry. It extends by Theorem 1 to an isometry  $\Delta_W \rightarrow \Delta$  and hence an apartment containing  $c, c'$ .) So the  $W$ -metric on  $\Delta$  can be recovered from the metric on the Coxeter complex; moreover, the metrics on overlapping Coxeter complexes agree on the overlaps:

If chambers  $c, c' \in A$  and  $c, c' \in A'$  then there is an isometry  $A \rightarrow A'$  fixing  $A \cap A'$ . (10)

(We leave this to the reader with the following hints: use the apartments to get an isometry  $A \rightarrow A'$  fixing a chamber  $c_0 \in A \cap A'$ ; then show that every chamber in the intersection is fixed by showing that in an apartment there is a unique chamber a given  $W$ -distance from  $c_0$ .)

It turns out that any chamber system covered by sufficiently many Coxeter complexes in a sufficiently nice way so that (9) and (10) hold can be made into a building by patching together the local metrics on the Coxeter complexes *ala* Lecture 1. To formulate this properly we need to replace isometries by maps not involving metrics. This is done in the following exercise.

*Exercise.* Let  $\Delta, \Delta'$  be chamber systems over the same set  $I$ . Show that (i).  $\alpha : (\Delta, \delta) \rightarrow (\Delta', \delta')$  is an isometry of buildings if and only if  $\alpha : \Delta \rightarrow \Delta'$  an injective morphism of chamber systems, and (ii).  $\alpha$  is a surjective isometry of buildings if and only if  $\alpha$  an isomorphism of chamber systems.

**Theorem 2.** Let  $(W, S)$  be a Coxeter group with  $S = \{s_i\}_{i \in I}$  and  $\Delta$  a chamber system over  $I$ . Suppose  $\Delta$  contains a collection  $\{A_\alpha\}$  of sub-chamber systems over  $I$ , called apartments, with each subsystem isomorphic (as a chamber system) to the Coxeter complex  $\Delta_W$ . Suppose also that

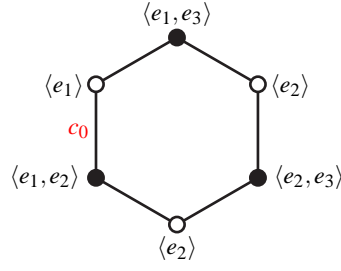
- (i). any two chambers  $c, c'$  of  $\Delta$  are contained in some apartment  $A$ , and
- (ii). if chambers  $c, c' \in A_\alpha$  and  $c, c' \in A_\beta$ , then there is an isomorphism  $A_\alpha \rightarrow A_\beta$  fixing  $A_\alpha \cap A_\beta$ .

Define  $\delta : \Delta \times \Delta \rightarrow W$  by  $\delta(c, c') = \delta_W(\alpha(c), \alpha(c'))$  where  $\alpha : \Delta_W \rightarrow A$  is an isomorphism with  $c, c' \in A$ . Then  $(\Delta, \delta)$  is a building of type  $(W, S)$ .

*Example 10 (the flag complex of Lecture 1 revisited).* The chamber system structure on the flag complex  $\Delta$  of Lecture 1 was given there (and in Example 6, where we saw that  $\Delta$  is thick). If  $L_1, L_2, L_3$  are lines in  $V$  spanned by independent vectors, then we get a hexagonal configuration as in Lecture 1. Let the apartments be all the hexagons obtained in this way. If  $c = V_1 \subset V_2$  and  $c' = V'_1 \subset V'_2$  are chambers, then they can be situated in an apartment by extending  $V_1, V'_1$  to a set  $L_1, L_2, L_3$  of independent lines. If  $V_1 \neq V'_1, V_2 \neq V'_2$  and  $V_2 \cap V'_2$  is a line different from  $V_1, V'_1$  as for the  $c, c'$  of Lecture 1, then this extension is unique, so  $c, c'$  lie in a unique apartment. Otherwise (e.g. if  $V_2 \cap V'_2$  is one of  $V_2$  or  $V'_2$ ) there is some choice. In any case, if  $L_1, L_2, L_3$  and  $L'_1, L'_2, L'_3$  are two such extensions corresponding to apartments  $A_\alpha, A_\beta$  containing  $c, c'$ , then any  $g \in GL(V)$  with  $g(L_i) = L'_i$  induces an isomorphism  $A_\alpha \rightarrow A_\beta$  that fixes  $A_\alpha \cap A_\beta$ .

## Lecture 5: Spherical Buildings

So far our supply of *thick* buildings is a little disappointing: only the flag complex of Lecture 1 and the affine building of Example 9. In this lecture we considerably increase the library by extracting a building from the structure of a reductive algebraic group. These guys really are the motivating examples of buildings.

Fig. 2. Apartment  $A_0$ 

Call a building of type  $(W, S)$  *spherical* when the Coxeter group  $(W, S)$  is spherical (i.e. finite). It turns out that there is a uniform construction of a large class of thick spherical buildings. To motivate this we reconstruct the flag complex building  $\Delta$  of Lecture 1 inside the general linear group  $G = GL(V) \cong GL_3(k)$ .

First, let  $P \subset G$  be the subgroup of permutation matrices – those matrices with exactly one 1 in each row and column and all other entries 0; alternatively, the  $a_\pi = \sum_j e_{\pi(j),j}$ , where  $\pi \in \mathfrak{S}_3$  and  $e_{ij}$  is the  $3 \times 3$  matrix with a 1 in the  $ij$ -th position and 0's elsewhere. The map  $\pi \mapsto a_\pi$  is an isomorphism  $\mathfrak{S}_3 \rightarrow P$  with

$$s_1 = (1, 2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } s_2 = (2, 3) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (11)$$

For the rest of the lecture we will blur the distinction between the symmetric group  $\mathfrak{S}_3$ , the group of permutation matrices  $P$ , and the Coxeter group  $(W, S)$  with the symbol  $\circ \text{---} \circ$ .

Assume for the moment that:

- (G1). The action of  $G$  on the flag complex  $\Delta$  given by  $g : V_1 \subset V_2 \mapsto gV_1 \subset gV_2$  is by chamber system isomorphisms (hence via isometries by the Exercise of Lecture 4).
- (G2). Fix  $g \in (W, S)$  and let  $X(g) = \{(c, c') \in \Delta \times \Delta \mid \delta(c, c') = g\}$ . Then for any  $g$  the diagonal action of  $G$  on  $X(g)$  is transitive (thus  $G$  acts transitively on the ordered pairs of chambers a fixed  $W$ -distance apart).
- (G3). Let  $A_0 \subset \Delta$  be the apartment given by the lines  $L_i = \langle e_i \rangle$  with  $\{e_1, e_2, e_3\}$  the usual basis for  $V$ , and  $c_0$  the chamber  $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle$  – see Figure 2. Then  $P$  acts on  $A_0$ . Moreover, the isometry  $\Delta_W \rightarrow A_0$ ,  $g \mapsto gc_0$  is equivariant with respect to the  $(W, S)$ -action  $g \xrightarrow{g_0} g_0g$  on the Coxeter complex  $\Delta_W$  and the  $P$ -action on the apartment  $A_0$  (thus, the  $(W, S)$ -action on  $\Delta_W$  is the same as the  $P$ -action on  $A_0$ ).

These allow us to reconstruct the chambers, incidences and  $\mathfrak{S}_3$ -metric:

*Reconstructing the chambers of  $\Delta$  in  $G$ .* For  $g \in G$  we have  $gc_0 = c_0$  with  $c_0 = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle$ , exactly when

$$g \in B := \left\{ \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \in G \right\},$$

the subgroup of upper triangular matrices. It is easy to show that (G2) is equivalent to (G2a): the  $G$ -action on  $\Delta$  is transitive on the chambers, and (G2b): for any  $g \in (W, S)$  the action of the subgroup  $B$  is transitive on the chambers  $c$  such that  $\delta(c_0, c) = g$ .

Combining (G2a) with the fact that the chamber  $c_0$  has stabilizer  $B$ , we get a 1-1 correspondence between the chambers of  $\Delta$  and the left cosets  $G/B$ :

$$\text{chambers } gc_0 \in \Delta \xleftrightarrow{1-1} \text{cosets } gB \in G/B.$$



*Reconstructing the  $i$ -incidences.* Let  $c_1, c_2 \in \Delta$  be 1-incident chambers:  $c_1 = V_1 \subset V_2$  and  $c_2 = V'_1 \subset V_2$ , and let  $c_i = g_i c_0$  with the  $g_i \in G$ . Then  $g_1^{-1} g_2$  stabilizes the subspace  $\langle e_1, e_2 \rangle$ , hence

$$g_1^{-1} g_2 \in \left\{ \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \in G \right\}. \quad (12)$$

*Exercise.* For  $s_1$  the permutation matrix in (11), show that the subgroup of matrices in (12) is the disjoint union  $B\langle s_1 \rangle B := B \cup B s_1 B$ , where  $BgB = \{bgb' \mid b, b' \in B\}$  is a double coset.

Returning to the incidences, if we are to replace the chambers  $c_1, c_2$  by the cosets  $g_1 B, g_2 B$ , then we need to replace  $c_1 \sim_1 c_2$  by  $g_1^{-1} g_2 \in B\langle s_1 \rangle B$ . Similarly

$$c_1 \sim_2 c_2 \text{ exactly when the } c_i = g_i c_0 \text{ with } g_1^{-1} g_2 \in \left\{ \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & \bullet & \bullet \end{pmatrix} \in G \right\} = B\langle s_2 \rangle B.$$

*Reconstructing the  $\mathfrak{S}_3$ -metric  $\delta$ .* Let  $c_1, c_2 \in \Delta$  be chambers with  $c_i = g_i c_0$ . Suppose that  $\delta(c_1, c_2) = g \in (W, S)$ . As  $G$  is acting by isometries (G1), we have  $\delta(c_0, g_1^{-1} g_2 c_0) = g$ . In the Coxeter complex  $\Delta_W$  we have by (8) that  $\delta_W(1, g) = g$ , so that by (G3),  $\delta(c_0, g c_0) = g$  also. Thus by (G2b) there is a  $b \in B$  with  $(bc_0, bgc_0) = (c_0, g_1^{-1} g_2 c_0)$ , so in particular,  $bgc_0 = g_1^{-1} g_2 c_0$ . As the elements of  $G$  sending  $c_0$  to  $bgc_0$  are precisely the coset  $BgB$ , we get  $g_1^{-1} g_2 \in BgB \subset BgB$ .

Conversely, if  $g_1^{-1} g_2 \in BgB$  then

$$\delta(c_1, c_2) = \delta(g_1 c_0, g_2 c_0) = \delta(c_0, g_1^{-1} g_2 c_0) = \delta(c_0, bgb' c_0) = \delta(c_0, bgc_0),$$

for some  $b \in B$ , so that

$$\delta(c_0, bgc_0) = \delta(bc_0, bgc_0) = \delta(c_0, gc_0) = \delta_W(1, g) = g,$$

(the first as  $B$  stabilizes  $c_0$ , the second by (G1) and the third by (G3)). We conclude that  $\delta(c_1, c_2) = g$  iff  $g_1^{-1} g_2 \in BgB$ .

Summarizing, let the left cosets  $G/B$  be a chamber system over  $I = \{1, 2\}$  with incidence defined by  $g_1 B \sim_i g_2 B$  iff  $g_1^{-1} g_2 \in B\langle s_i \rangle B$  and  $\mathfrak{S}_3$ -metric  $\delta(g_1 B, g_2 B) = g$  iff  $g_1^{-1} g_2 \in BgB$ . Then  $G/B$  is a building of type  $\bigcirc \text{---} \bigcirc$ , isomorphic to the flag complex of Lecture 1.

*Exercise.* Show that the assumptions (G1)-(G3) hold (*hint*: for (G2) with  $\delta(c_1, c_2) = \delta(c'_1, c'_2)$ , situate  $c_1, c_2$  in a hexagon as in Lecture 1 and  $c'_1, c'_2$  similarly. Then use the fact that  $GL_3(k)$  acts transitively on ordered bases of  $V$ ).

We are feeling our way towards a class of groups in which we can mimic this reconstruction of the flag complex. It turns out to be convenient to formulate the class abstractly first, and then bring in the natural examples later.

A *Tits system* or *BN-pair* for a group  $G$  is a pair of subgroups  $B$  and  $N$  of  $G$  satisfying the following axioms:

(BN0).  $B$  and  $N$  generate  $G$ ;

(BN1). the subgroup  $T = B \cap N$  is normal in  $N$ , and the quotient  $N/T$  is a Coxeter group  $(W, S)$  for some  $S = \{s_i\}_{i \in I}$ ;

(BN2). for every  $g \in W$  and  $s \in S$  the product of double cosets<sup>2</sup>  $BsB \cdot BgB \subset BgB \cup BsgB$ ;

<sup>2</sup> A  $g \in W$  is not an element of  $G$  but a coset  $\bar{g}T$ . As  $T \subset B$ , if  $\bar{g}_1 T = \bar{g}_2 T$  then  $B\bar{g}_1 B = B\bar{g}_2 B$ , so we can unambiguously write  $BgB$  to mean  $B\bar{g}B$  for some representative in  $N$  for  $g$ .

(BN3). for every  $s \in S$  we have  $sBs \neq B$ .

The group  $W$  is called the *Weyl group* of  $G$ , and is in general not finite.

*Example 11.*  $G = GL_n(k)$ ;  $B$  = the upper triangular matrices in  $G$ ;  $N$  = the monomial matrices in  $G$  (those having exactly one non-zero entry in each row and column),

$$T = \{\text{diag}(t_1, \dots, t_n) \mid t_1 \dots t_n \neq 0\},$$

and  $W$  = the permutation matrices with

$$s_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & \boxed{\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}} & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

for  $i \in \{1, \dots, n-1\}$  where there are  $i-1$  1's on the diagonal before the  $2 \times 2$  block. Let  $e_i$  be the  $n$ -column vector  $(0, \dots, 1, \dots, 0)^T$  with the 1 in the  $i$ -th position and  $L_i = \{te_i \mid t \in k\}$ . Then  $N$  permutes the set of lines  $\{L_1, \dots, L_n\}$  and  $W$  is isomorphic to the symmetric group on this set (hence  $\cong \mathfrak{S}_n$ ). This example is misleadingly special in that the extension  $1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1$  splits, so that the Weyl group  $W$  can be realised, via the permutation matrices, as a subgroup of  $G$ . In general this doesn't happen.

**Theorem 3.** Let  $G$  be a group with a BN-pair and let  $\Delta$  be a chamber system over  $I$  with chambers the cosets  $G/B$  and incidence defined by  $g_1B \sim_i g_2B$  iff  $g_1^{-1}g_2 \in B\langle s_i \rangle B$ . Define a  $W$ -metric by  $\delta(g_1B, g_2B) = g \in W$  iff  $g_1^{-1}g_2 \in BgB$ . Then  $(\Delta, \delta)$  is a thick building of type  $(W, S)$ .

*Example 12.*  $G$  = the symplectic group  $Sp_{2n}(k) = \{g \in GL_{2n}(k) \mid g^T J g = J\}$  where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

with  $I_n$  the  $n \times n$  identity matrix;  $B$  = the upper triangular matrices in  $Sp_{2n}(k)$ ;  $N$  = the monomial matrices in  $Sp_{2n}(k)$ , and

$$T = \{\text{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}) \mid t_i \neq 0\}.$$

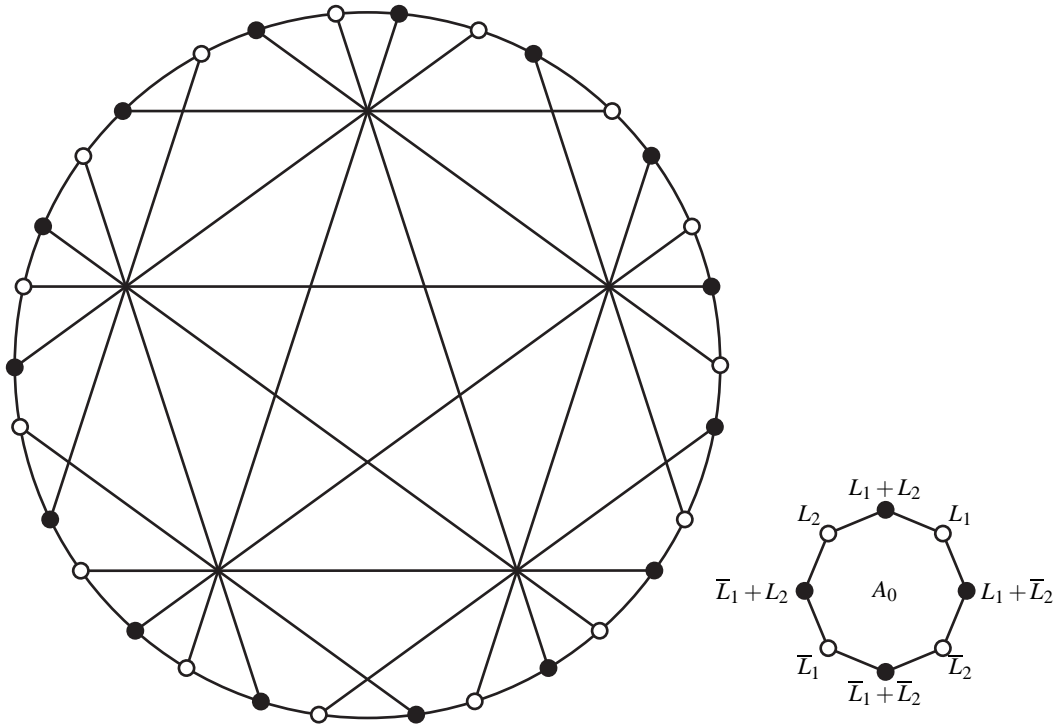
For  $1 \leq i \leq n$  let  $e_i$  be the  $2n$ -column vector  $(0, \dots, 1, \dots, 0)^T$  with the 1 in the  $i$ -th position and  $\bar{e}_{i-n}$  similarly, for  $n+1 \leq i \leq 2n$ . Let  $L_i = \{te_i \mid t \in k\}$  and  $\bar{L}_i = \{t\bar{e}_i \mid t \in k\}$ , writing  $\bar{L} = L$ . Then  $N$  permutes the set  $\{L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n\}$  and  $W$  is isomorphic to the “signed” permutations  $\mathfrak{S}_{\pm n} = \{\pi \in \mathfrak{S}_{2n} \mid \pi(\bar{L}_i) = \overline{\pi(L_i)}\}$ .

*Exercise.* Let  $V$  be a  $2n$ -dimensional space over  $k$  and  $(u, v)$  a symplectic form on  $V$  – a non-degenerate alternating bilinear form<sup>3</sup>. Let  $O(V)$  be those linear maps preserving the form, i.e.  $O(V) = \{g \in GL(V) \mid (g(u), g(v)) = (u, v) \text{ for all } u, v \in V\}$ . The form can be defined on a basis  $\{e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n\}$  by

$$(e_i, e_j) = 0 = (\bar{e}_i, \bar{e}_j) \text{ and } (e_i, \bar{e}_j) = \delta_{ij} = -(\bar{e}_j, e_i),$$

so that  $O(V) \cong Sp_{2n}(k)$ . Call a subspace  $U \subset V$  totally isotropic if  $(u, v) = 0$  for all  $u, v \in U$ . It turns out that the maximal totally isotropic subspaces are  $n$ -dimensional. A (maximal) flag in  $V$

<sup>3</sup> Alternating means  $(u, u) = 0$  for all  $u$ , and non-degenerate that  $V^\perp = \{0\}$



**Fig. 3.** The spherical building of the symplectic group  $Sp_4(\mathbb{F}_2)$  and apartment  $A_0$ .

is a sequence of totally isotropic subspaces  $V_1 \subset \cdots \subset V_n$  with  $\dim V_i = i$ . Let  $\Delta$  be the chamber system with chambers these flags and incidences over  $I = \{1, \dots, n\}$  as in the flag complex of Example 6:  $(V_1 \subset \cdots \subset V_n) \sim_i (V'_1 \subset \cdots \subset V'_n)$  when  $V_j = V'_j$  for  $j \neq i$ . Let  $c_0$  be the chamber

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \dots, e_n \rangle$$

and  $A_0$  the set of images of  $c_0$  under the signed permutations  $\mathfrak{S}_{\pm n} = \{\pi \in \mathfrak{S}_{2n} \mid \pi(\bar{e}_i) = \overline{\pi(e_i)}\}$  (writing  $\bar{e} = e$ ). Finally, let  $\{A_\alpha\}$  be the set of images of  $A_0$  under  $Sp_{2n}(k)$ . Show that with this set of apartments  $\Delta$  is a building isomorphic to the spherical building of  $Sp_{2n}(k)$  arising from Theorem 3 and Example 12.

We finish where we started by drawing a picture. Let  $V$  be 4-dimensional over the field of order 2 and equipped with symplectic form  $(u, v)$ . Let  $\Delta$  be the graph with vertices the proper non-trivial totally isotropic subspaces of  $V$ , with an edge connecting the (white) 1-dimensional vertex  $V_i$  to the (black) 2-dimensional vertex  $V_j$  whenever  $V_i$  is a subspace of  $V_j$ . Any 1-dimensional subspace (of which there are fifteen) is totally isotropic, and is contained in three 2-dimensional totally isotropic subspaces, each of which in turn contains three 1-dimensional subspaces. There are thus fifteen 2-dimensional vertices. The local pictures/apartments are octagons (or barycentrically subdivided diamonds). The apartment  $A_0$  of the Exercise above has white vertices  $L_1, L_2, \bar{L}_1, \bar{L}_2$ , using the notation of Example 12, and black vertices  $L_1 + L_2, L_1 + \bar{L}_2, \bar{L}_1 + L_2$  and  $\bar{L}_1 + \bar{L}_2$ . See Figure 3.

*Remark.* Examples 11 and 12 are of groups contained in classical groups of matrices. This can be generalized. Let  $k = \bar{k}$  be algebraically closed and  $G$  a connected algebraic group defined over  $k$ . Suppose also that  $G$  is reductive, i.e. that its unipotent radical is trivial. Let  $B$  be a Borel subgroup (a maximal closed connected soluble subgroup) and  $T \subset B$  a maximal torus – a subgroup isomorphic to  $(k^\times)^m$  for some  $m$ . Finally, let  $W = N/T$  be the Weyl group of  $G$ ,

where  $N$  is the normalizer in  $G$  of  $T$ . This is isomorphic to a *finite* Coxeter group  $(W, S)$  with  $S = \{s_i\}_{i \in I}$ . The result is a  $BN$ -pair for  $G$ . For a general non-algebraically closed  $k$  a  $BN$ -pair can still be extracted from  $G$ , but one has to tread more carefully.

## Notes and References

*Lecture 1.* This is mostly folklore. The reader is to be minded of projective geometry as  $\Delta$  is the incidence graph of the standard projective plane over  $k$ . The ad-hoc argument (essentially the Jordan-Hölder Theorem) for associating the permutation  $(1, 3)$  to the pair of chambers is from [AB08, §4.3].

*Lecture 2.* Standard references on reflection groups and Coxeter groups are [Bou02] (still the only place you can find some things), [Hum90] and [Kan01]. The definition of reflection in (1) is from [Bou02, V.2.2]. That  $\mathcal{H}$  consists of all the reflecting hyperplanes of  $W$  (the claim at the end of the Exercise) is [Hum90, Proposition 1.14]. The general theory of finite reflection groups, including their classification, is to be found in Chapters 1 and 2 of [Hum90]. Example 3, although fairly standard, is taken from [AB08, 2.2.2]. The general theory of affine groups is in [Hum90, Chapter 4]. For the hyperboloid or Minkowski model of hyperbolic space, hyperbolic lines, etc, see [Rat06, Chapter 3]. The standard reference on hyperbolic reflection groups is [Vin85]. The treatment of chambers, panels and adjacency is taken from [AB08, 1.1.4]. That  $W$  acts regularly on the chambers is [Hum90, Theorem 1.12]. Fact 1 is [Hum90, Theorem 1.5] and Fact 2 is [Hum90, Theorem 1.9]. Coxeter groups as a notion are generally attributed to Tits, and while I'm not 100% sure, they probably first appeared in the original 1968 edition of [Bou02, IV.1]; a good place to start is [Hum90, Chapter 5]. The name is a homage to [Cox35]. The representation  $(W, S) \rightarrow GL(V)$  described in the final remark is called the geometric or reflectional or Tits representation. It is defined in [Hum90, 5.3] and faithfulness is [Hum90, Corollary 5.4] or [AB08, Theorem 2.59], where it is shown that the image of  $(W, S)$  is discrete.

*Lecture 3.* Apart from the aside, this lecture is based mainly on Chapters 1-2 of [Ron09]; the initial chamber system notions and Example 6 are directly from [Ron09, §1.1]. Chapter 2 of this book is entirely devoted to Coxeter complexes. A thorough exploration of the general connections between chambers systems and simplicial complexes is given in [AB08, Appendix A]. The building specific set-up is in [AB08, §5.6]. The construction of the simplicial complex  $X_\Delta$  as the nerve of the covering by rank  $|I| - 1$  residues is [AB08, Exercise 5.98]. The statement about the intersection of residues being a residue is [AB08, Exercise 5.32].

*Lecture 4.* This lecture is almost entirely based on Chapter 3 of [Ron09] from which the definition of building is taken. That the Coxeter complexes comprise the thin buildings is from [Ron09, §3.2]. A theorem of Walter Feit and Graham Higman [FH64] has consequence that a finite thick building has type  $(W, S)$  a finite reflection group where each irreducible component of  $W$  is of type  $A_n, B_n/C_n, D_n, E_6, E_7, E_8, F_4, G_2$  or  $I_2(8)$  (see [AB08, Theorem 6.94]; see [Hum90, Chapter 2] for a description of these types of finite reflection group). Hence the statement about the symmetry group of the dodecahedron, for which  $(W, S)$  has type  $H_3$ . Example 9 is a special case of a construction that extracts a  $BN$ -pair, and hence an affine building, from an algebraic group defined over a field with a discrete valuation. Example 9 is thus the affine building for  $SL_2\mathbb{Q}_2$  with  $\mathbb{Q}_2$  the 2-adics. The fact that for any prime  $p$  the affine building for  $SL_2\mathbb{Q}_p$  is a tree was used by Serre to reprove a theorem of Ihara that a torsion free lattice in  $SL_2\mathbb{Q}_p$  is a free group [Ser03, II 1.1]. For affine buildings in general see the book of Weiss [Wei09]. Theorem 1 is [Ron09, Theorem 3.6] and Theorem 2 is [Ron09, Theorem 3.11].

*Lecture 5.* This lecture is based on [Ron09, Chapter 5]. Properties (G1)-(G3) are the specialization to  $GL_3$  of a strongly transitive group action [Ron09, §5.1]. The argument that reconstructs the  $W$ -metric is taken from the proof of [Ron09, Theorem 5.2]. The axioms for a  $BN$ -pair are from [Ron09, §5.1]. A proof that Example 11 is a  $BN$ -pair using nothing but row and column operations can be found in [AB08, §6.5]. Theorem 3 is [Ron09, Theorem 5.3]. The source is [Tit74]. The flag complex of a symplectic space exercise is from [Ron09, Chapter 1]. Figure 3 has several names: in graph theory circles it is called Tutte's eight-cage, and is the unique smallest cubic graph with girth 8 (where these minimal 8-circuits are, of course, the apartments). It is a pleasantly mindless exercise to label the vertices of the Figure with the totally isotropic subspaces (*hint*: start with the 8-circuit at the top as the apartment  $A_0$ ). That the  $B$  (Borel subgroup) and  $N$  (normalizer of a maximal torus) extracted from a reductive group  $G$  are a  $BN$ -pair for  $G$  is shown in [Hum75, §29.1].

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